EPFL– Fall 2025
**Series 3**

# Differential Geometry I: Curves and surfaces

G. Moschidis
26 Sep. 2025

---

**3.1** Express the length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the form of an integral (do not attempt to calculate this integral, as it cannot be calculated by elementary means; this integral gave rise to the theory of elliptic functions).

**3.2** The *astroid* is the plane curve with equation

$$|x|^{\frac{2}{3}} + |y|^{\frac{2}{3}} = 1.$$

**(a)** Draw the astroid.

**(b)** Find a parametrization of the astroid. What are the singular points?

**(c)** Compute the length of one cycle of the astroid.

**(d)** Compute the arc length parameter for the part of the curve in the first quadrant with initial point at $(1,0)$.

**(e)** Find the natural parametrization for the part of the curve in the first quadrant with the same initial point.

**3.3** Let $\gamma : I \to \mathbb{R}^n$ be a twice differentiable regular curve with constant speed. Show that at every point of $\gamma$, the acceleration is perpendicular to the velocity.

**3.4** **(a)** Let $(x,y)$ be Cartesian coordinates on $\mathbb{R}^2$. Recall the precise definition of polar coordinates $(r,\theta)$ and specify their domain of definition.

**(b)** Write the general equation of a line in polar coordinates, then the equation of a circle of radius $a$ and center $c = (r_0, \theta_0)$.

**(c)** Let $\gamma(t) = (r(t), \theta(t))$ be a $C^1$ curve written in polar coordinates. Find and prove a formula giving its length in these coordinates.

**(d)** The logarithmic spiral is the plane curve with polar equation $r = e^{\theta}$. Use the previous formula to compute the length of one cycle of this spiral defined by $0 \leqslant \theta \leqslant 2\pi$. Then find the natural parametrization with initial point $(1,0)$.

# Differential Geometry I: Curves and surfaces

**3.5** The *conchoid of Nicomedes* (belonging to the broader family of conchoid curves) is the curve $\mathcal{C}$ in the Euclidean plane defined as follows: Fix a $b \geqslant 0$, a point $O$ in the plane (called the *pole*) and a line $D$ (called the *directrix*) not passing through $O$. For any point $p$ in the plane such that $p \notin D$ and $p \neq O$, let

$$f(p) = d(p, q),$$

where $q$ is the intersection of $D$ with the line through $O$ and $p$, i.e.

$$q = (O + \mathbb{R}\,\overrightarrow{Op}) \cap D.$$

Then the curve $\mathcal{C}$ is defined as the level set

$$\mathcal{C} = \{p \in \mathbb{R}^2 \mid f(p) = b\}.$$

(a) Draw the curve $C$. Is it connected?

(b) Give a polar equation of this curve (assume that the directrix is vertical and that the pole is the origin).

(*c) Nicomedes ($3^{rd}$ century BC) used this curve to solve the problem of trisection of an angle (which is of course not possible just with a ruler and compass). This was his process: Suppose that $\widehat{AOB} = \theta_0$ is the angle we want to trisect (and assume that $\theta_0 < \frac{\pi}{2}$).

- $*$ Draw a line $m$ perpendiculat to $OA$. Let $P$ be the point of interection of $m$ with $OB$.
- $*$ Let $\mathcal{C}$ be the conchoid curve with pole $O$, directrix $m$ and $b = 2d(O, P)$. Keep the connected component of the curve that lies in the half plane defined by $m$ not containing $O$.
- $*$ Draw a line from $P$ which is parallel to $OA$. Let $M$ be the intersection point with $\mathcal{C}$.

Then $\widehat{AOM} = \frac{1}{3}\theta_0$. Prove this statement.

**3.6** Let $F : I \to SO(n) \subset \mathcal{M}_n(\mathbb{R}) = \mathbb{R}^{n\times n}$ be a $C^1$ curve taking values in the orthogonal group. Prove that $F(t)^{-1}\dot{F}(t)$ and $\dot{F}(t)F(t)^{-1}$ are skew-symmetric matrices for all $t \in I$. *(Recall that a matrix $A \in \mathcal{M}_n(\mathbb{R})$ is skew symmetric if $A^T = -A$.)*

**3.7** Recall that the exponential $\exp(A)$ of a square matrix $A \in M_n(\mathbb{R})$ is defined by the series

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!}A^k = I + A + \frac{1}{2!}A^2 + \cdots$$

This series is known to converge. It is also known that if $AB = BA$, then $\exp(A + B) = \exp(A)\exp(B)$ (the proof is the same as for the exponential of a sum of real numbers).

**(a)** Show that if $A \in M_n(\mathbb{R})$ is antisymmetric, then $\exp(A) \in SO(n)$.

**(b)** Compute the matrix $\exp(tJ)$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

**3.8** Prove the following statement or find a counterexample: If $\gamma_n : [a,b] \to \mathbb{R}^n$ is a sequence of curves converging *uniformly* to the curve $\gamma : [a,b] \to \mathbb{R}^n$ (all curves are assumed to be of class $C^1$), then the lengths converge, i.e.

$$\ell(\gamma) = \lim_{n\to\infty} \ell(\gamma_n).$$

**3.9** The goal of this exercise is to define the notion of intrinsic distance in a domain of $\mathbb{R}^n$ (by definition, a domain of $\mathbb{R}^n$ is an open, connected subset).

Let $\mathcal{U} \subset \mathbb{R}^n$ and $p, q \in \mathcal{U}$. Let $\mathcal{C}_{pq}$ denote the set of curves $\gamma : [a,b] \to \mathcal{U}$ that are continuous, piecewise $C^1$, and connect $p$ to $q$. The intrinsic distance in $\mathcal{U}$ from $p$ to $q$ is then defined by

$$\delta_{\mathcal{U}}(p,q) = \inf\{\ell(\gamma) \mid \gamma \in \mathcal{C}_{pq}\}.$$

- **(a)** Prove that $\mathcal{C}_{pq} \neq \varnothing$ for all $p, q \in \mathcal{U}$.
- **(b)** Prove that $\delta_{\mathcal{U}}(p,q) \geqslant \|q-p\|$ for all $p, q \in \mathcal{U}$.
- **(c)** Prove that $(\mathcal{U}, \delta_{\mathcal{U}})$ is a metric space.
- **(d)** Under what condition on the domain $\mathcal{U}$ do we have $\delta_{\mathcal{U}}(p,q) = \|q-p\|$ for all $p, q \in \mathcal{U}$? (give a sufficient condition).
- **(e)** Consider the case $\mathcal{U} = \{(x,y) \in \mathbb{R}^2 \mid x < -1 \text{ or } y \neq 0\}$. What is the intrinsic distance between $p = (0,1)$ and $q = (0,-1)$? Does there exist a curve of minimal length connecting $p$ to $q$?